On Application of Set Theory to Some Aspects of Real Analysis and Topology

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Authors' contributions

This work was carried out in collaboration between both authors. The corresponding author ECE designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. The co-author OGU managed the analyses of the study and managed the literature searches. Both authors read and approved the final manuscript.

Abstract

Set as a collection of mathematical elements makes more meaning in classical analysis and topology. There are diverse forms of its existence and operations. In the course of this work, we discussed its various operations such as union intersection complementation and its various algebras. The application of the various algebra in real analysis and metric space topology was peripherally discussed. Conclusively, we discovered that set is an all round player in analytical and topology mathematics.

Keywords: Function; metric space; partial ordering; relation; set; sets of real numbers.

1 Introduction

Definition of Set:

The set $A$ [1] is the set of elements $x \in X$ with the property $A = \{x \in A : p(x)\}$ thus $x \in A \iff [x \in X \text{ and } P(x)]$ provided the set $X$ is understood.

1.1 Types of set

1. Empty Set: Denoted by $\emptyset$, note $\emptyset \subseteq A \Rightarrow$ the empty set is a proper subject of the set $A$.
2. Singleton: The set $\{x\}$ is the singleton or unit set
3. Ordered Pair: $(x, y) = (a, b)$ if $x = a$, and $y = b$ so that $(x, y) \neq (y, x)$ if $x \neq y$.
4. Similarly defined is the ordered triple $(x, y, t)$.

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An unordered pair comes into play when there is no preference given to \(x\) and \(y\) i.e. \((x, y) = (y, x)\)

5. **Cartesian Product:** If \(X\) and \(Y\) are two sets, we define the Cartesian or direct product \(X \times Y\) to be the set \(\{(x, y)\}\) of all ordered pairs whose first element belongs to \(X\) similarly \(X \times Y \times T\) is the set \(\{(x, y, t)\}\) of all triples with that \(x \in X, y \in Y\) and \(t \in T\). If \(X\) is the set of real numbers, then \(X \times X\) is the set of ordered pairs of real numbers. We can sometimes write \(X^2\) for \(X \times X\) and so on.

1.2 Functions

A function \(f: X \to Y\) [1] is a rule that assigns to each element \(x\) in \(X\) a unique element \(f(x)\) in \(Y\). The collection \(C\) of pairs \((x, f(x))\) in \(X \times Y\) is called the graph of \(f\) and the subset \(C\) of \(X \times Y\) is called the graph if and only if for each \(x \in X\) such that a unique pair in \(C\) whose element is \(x\). We note that the set \(X\) is called the domain or domain of definition of \(f\). The set \(\{y \in Y; (t, x)[y = f(x)]\}\) is called the range of \(f\) which is smaller than \(Y\), but if the range of \(f\) is \(y\) then \(f\) is onto (surjective). The image of \(f\) is defined as \(f[A] = \{y \in Y; f(x)[x \in A\} and the range of \(f\) is \(f[x]\) and \(f\) is onto if \(f\) given, \(y = f(x)\). If \(B\) is a subset of \(Y\), then \(f^{-1}(B)\) of \(B\) is the set of those \(x \in X\) for which \(f(x)\) is on \(B\) i.e. \(f^{-1}[B] = \{x \in X; f(x) \in B\}\). We must note that \(f\) is onto \(y\) if \(f\) is the inverse image of each nonempty set of \(y\) is nonempty.

A function \(f: X \to Y\) is one to one (or injective or univalent) if \(f(x_1) = f(x_2)\) only when \(x_1 = x_2\) and any function that satisfies that condition is said to be in one to one correspondence between \(X\) and \(Y\) (they are also called bijective) in this case if there is a function \(g: Y \to X\) such that for all \(x, y\) we have \(g(f(x)) = x\) and \(f(g(y)) = y\) where \(g\) is the inverse of \(f\) and is denoted by \(f^{-1}\). If \(g = f^{-1}\), then \(f^{-1}[\varepsilon]\) can be considered to be the inverse image of \(\varepsilon\) under \(f\) or the image of \(\varepsilon\) under \(f^{-1}\) and these are the same set.

If \(f: X \to Y\) and \(g: Y \to Z\), we define a new function \(h: X \to Z\) by setting \(h(x) = g(f(x))\) where \(h\) is called the composition of \(g\), setting \(h(x) = g(f(x))\), the function \(h\) is called the composition of \(g\) with \(f\) and denoted by \(g\). If \(f: X \to Y\) and \(A\) is a subset of \(X\), we can define a new function \(g: A \to Y\) by defining \(g(x) = f(x)\) for \(x \in A\). This new function \(g\) is called the restriction of \(f\) to \(A\) and sometimes \(f/A\).

The difference between \(f\) and \(f/A\) is that they have different ranges and the inverse images under \(g\) are different from those under \(f\). Here we must note that ordered pairs may be defined in the writing of a function that is an ordered pair of a function whose domain is in the set \(\{1, 2\}\) and similarly on finite sequence, or \(a\)-triple is a function whose domain is the function of a natural number such that \(\{i \in N; i \leq n\}\). Such a set is called a segment of \(N\). Similarly, an infinite sequence be it finite or infinite is a function whose domain is the set of natural numbers if a sequence is in a set \(X\), we peak of a sequence from or in \(X\) or of a sequence of elements of \(X\).

Then we can denote by \(\{x_i\}_{i=1}^n\) for finite sequence and \(\{x_i\}_{i=1}^\infty\) for infinite sequence the range of the sequence we denote by \(\{x_i\}\). Thus the range of an ordered \(n\)-triple \(\{x_i\}_{i=1}^n\) is the unordered \(n\)-triple \(\{x_i\}_{i=1}^n\) which is consistent with our earlier notation concerning ordered and unordered triples and so forth.

A set is called countable if it is the range of some sequence and finite if it is the range of some finite sequence. A set that is not finite is called infinite note that countable infinite sets means infinite or defined infinite sequence.

1.3 Principle of recursive set

**Definition 1.3.1:** [2] Let \(f\) be function from a set \(X\) to itself and let \(a\) be an element of \(X\), then there is a unique infinite sequence \(\{x_i\}\) from \(X\) such that \(x_i = a\) and \(x_i + 1 = f(x_i)\) for each \(i\).
The empty set
And we also have relations between unions and intersections which in
Collection of pair wise disjoint sets of any two sets in
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We now introduce the notion of subsequence which is connected with sequence. We say that
Then there is a unique sequence \( \{x_i\} \) from \( X \) such that \( x_i = a \) and \( x_i + 1 = f(x_i,x_{i+1}) \).
We now introduce the notion of subsequence which is connected with sequence. We say that \( f \) is a \( n \) mapping \( g : N \to N \) which is monotone if \( (i > 1) \Rightarrow g(i) > g(i) \). If \( f \) is an infinite sequence that is a function whose domain is \( N \), we say that \( h \) is an infinite subsequence of \( f \) if there is a monotone mapping of \( N \) into \( N \) such that \( h = f \) if we wrote \( f \) as \( \{f_i\} \) and \( g \) as \( \{g_i\} \), then we usually denote \( f \circ g \) by \( \{f \circ g_i\} \).

1.4 Unions, intersections and complements

Let us fix a given set \( X \) and consider the set \( \mathcal{P}(X) \) consisting of all subsets of \( X \). If \( A \) and \( B \) are subsets of \( X \), we [3] define their intersection \( A \cap B \) to be the set of all elements that belong to both \( A \) and \( B \). Thus

\[
A \cap B = \{x: x \in A \& x \in B\}
\]

We note that the definition is symmetric in \( A \) and \( B \); that is \( A \cap B = B \cap A \). Also, \( A \cap B \subset A \) and \( A \cap B = A \leftrightarrow A \cap B \). We have \( (A \cap B) \cap C = A \cap (B \cap C) \) and write this to each of the sets \( A, B \) and \( C \). We define the union \( A \cup B \) of two sets \( A \) and \( B \) to be the set of elements that are in either \( A \) or \( B \). Thus,

\[
A \cup B = \{x: x \in A \cup x \in B\}
\]

We have

\[
A \cup B = B \cup A
\]

\[
A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C
\]

\[
A \subset A \cup B
\]

\[
A = A \cup B \iff B \subset A
\]

Collection of pair wise disjoint sets of any two sets in \( C \) are disjoint

\[
f^{-1}[\cap A B_i] = \cap f^{-1}[B_i]
\]

And we also have relations between unions and intersections which in [4] are called distributive laws:

\[
A \cap (B \cup C) = (A \cap B) \cup (A \cap C)
\]

\[
A \cup (B \cap C) = (A \cup B) \cap (A \cup C)
\]

The empty set \( \emptyset \) and the space \( X \) play special roles:

\[
A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset
\]

\[
A \cup X = X, \quad A \cap X = A
\]
If $A$ is a subset of $X$, we define the complement $\tilde{A}$ of $A$ (relative to $X$) as the set of elements not in $A$. Thus

$$\tilde{A} = \{ x \in X: x \notin A \}$$

We sometimes write $\sim A$ instead of $\tilde{A}$. We have

$$\emptyset = X, \quad \tilde{X} = \emptyset$$

$$\tilde{A} = A, \quad A \cup \tilde{A} = X, \quad A \cap \tilde{A} = \emptyset$$

$$A \subset B \Leftrightarrow \tilde{B} \subset \tilde{A}$$

Two special laws relating complements to unions and intersections are De Morgan’s laws:

$$\sim (A \cup B) = \tilde{A} \cap \tilde{B}$$

$$\sim (A \cap B) = \tilde{A} \cup \tilde{B}$$

If $A$ and $B$ are two subsets of $X$, we define the difference $B \sim A$, or relative complement of $A$ in $B$, as the set of elements in $B$ that are not in $A$. Thus

$$B \sim A = \{ x: x \in B \& x \notin A \}.$$  

We have $B \sim A = B \cap \tilde{A}$.

We shall also use the notation $A \Delta B$ for the symmetric difference of two sets defined by

$$A \Delta B = (A \sim B) \cup (B \sim A).$$

The symmetric difference of two sets consists of all those points that belong to one or the other of the two sets but not to both. If the intersection of two sets is empty, we say the sets are disjoint. A collection $C$ of sets is said to be a disjoint collection of sets or a collection of pairwise disjoint sets if any two sets in $C$ are disjoint.

$$f^{-1}[\cap_{A \in C} \tilde{A}] = \cap_{A \in C} f^{-1}[B_A]$$

And the process of taking unions (or intersections) of two sets can be extended by repetition to give unions/intersections of any finite collection of sets. However, we can give a definition of intersection for an arbitrary collection $\mathcal{C}$ of sets: the intersection of the collection $\mathcal{C}$ is the set of those elements of $X$ that belong to each member of $\mathcal{C}$. We denote this intersection by

$$\cap_{A \in \mathcal{C}} A \quad \text{or} \quad \cap \{ A: A \in \mathcal{C} \}$$

Similarly, we define the union of an arbitrary collection of sets by

$$\cup_{A \in \mathcal{C}} A = \{ x \in X: (\exists A)(A \in \mathcal{C} \Rightarrow x \in A) \}$$

De Morgan’s laws hold for arbitrary unions and intersections

$$\sim [\cup_{A \in \mathcal{C}} A] = \cap_{A \in \mathcal{C}} \tilde{A}$$
\[ \sim [\cap_{\Delta} E A] = \cup_{\Delta} E \bar{A} \]

We also have the distributive laws:

\[ B \cap [\cup_{\Delta} E A] = \cup_{\Delta} E (B \cap A) \]

\[ B \cup [\cap_{\Delta} E A] = \cap_{\Delta} E (B \cup A) \]

It follows from our definition that the union of an empty collection of sets is empty and that the intersection of the empty collection of sets is \( X \).

By a sequence of subsets of \( X \) we mean a sequence from \( \wp(X) \) that is, a mapping of \( N \) (or a segment of \( N \)) into \( \wp(X) \). If \( (A_i) \) is an infinite sequence of subsets of \( X \), we write \( \cup_{i=1}^{\infty} A_i \) for the union of the range of the sequence. Thus

\[ \cup_{i=1}^{\infty} A_i = \{ x : (\exists i)(x \in A_i) \} \]

Similarly, if \( (B_i)_{i=1}^{n} \) is a finite sequence of subsets of \( X \), we write \( \cap_{i=1}^{n} B_i \) for the intersection of the range of the sequence, so that

\[ f^{-1}[\cap_{\lambda} B_{\lambda}] = \cap_{\lambda} f^{-1}[B_{\lambda}] \]

and this notation for sequences of sets is so convenient that we often generalize it to arbitrary collections of sets by using the notion of an indexed collection: an indexed subset of \( \wp(X) \), or collection of subsets of \( X \) is a function on an index set \( \Lambda \) to \( X \) (or the set of subsets of \( X \)). If \( \Lambda \) is the set of natural numbers, then the notion of an indexed set coincides with the notion of a sequence.

In keeping with the notation for sequences, \([6]\) usually write \( x_{\lambda} \) instead of \( x(\lambda) \), and denote the indexed set itself by \( \{ x_{\lambda} \} \) or \( \{ x_{\lambda} : \lambda \in \Lambda \} \). We say that \( \{ x_{\lambda} \} \) is indexed by \( \Lambda \). We define the union and intersection of an indexed set to be the union and intersection of the range of the function defining the indexed set. Thus

\[ \cup_{\lambda \in \Lambda} A_{\lambda} = \{ x \in X : (\exists \lambda)(\lambda \in \Lambda \& x \in A_{\lambda}) \} \]

and

\[ \cup_{\lambda \in \Lambda} A_{\lambda} = \{ x \in X : (\lambda)(\lambda \in \Lambda \Rightarrow x \in A_{\lambda}) \} \]

In the case when \( \Lambda \) is the set \( N \) of natural numbers we have

\[ \cap_{i \in N} A_i = \cap_{i=1}^{\infty} A_i \]

and similarly for unions.

If \( f \) maps \( X \) into \( Y \) and \( \{ A_{\lambda} \} \) is a collection of subsets of \( X \), then

\[ f[\cup_{\lambda} A_{\lambda}] = \cup_{\lambda} f[A_{\lambda}] \]

But we can conclude only that

\[ f[\cap_{\lambda} A_{\lambda}] \subseteq \cap_{\lambda} f[A_{\lambda}] \]
For inverse images we have for a collection \( \{ B_\lambda \} \) of subsets of \( Y \),
\[
    f^{-1} [ \bigcup_\lambda B_\lambda ] = \bigcup_\lambda f^{-1} [ B_\lambda ]
\]
\[
    f^{-1} [ \bigcap_\lambda B_\lambda ] = \bigcap_\lambda f^{-1} [ B_\lambda ]
\]
and
\[
    f^{-1} [ f[A] ] \supset A
\]
For \( A \subset X \) and \( B \subset Y \)
\[
    f^{-1} [ \overline{B} ] = \overline{f^{-1} [ B ]}
\]
For \( B \subset Y \), also
\[
    f [ f^{-1} [ B ] ] \subset B
\]

1.5 Algebrae of sets

A collection \( \alpha \) of subsets of \( X \) is called an algebra of sets or a Boolean algebra if (i) \( A \cup B \) is in \( \alpha \) whenever \( A \) and \( B \) are, and (ii) \( \overline{A} \) is in \( \alpha \) whenever \( A \) is, it follows from De Morgans’ laws that (iii) \( A \cap B \) is in \( \alpha \) whenever \( A \) and \( B \) are. If a collection \( \alpha \) of subsets of \( X \) satisfies (i) and is therefore a Boolean algebra. By taking unions two at a time, we see that if \( A_1, \ldots, A_n \) are sets in \( \alpha \), then \( A_1 \cup A_2 \cup \ldots \cup A_n \) is again in \( \alpha \). Similarly, \( A_1 \cap A_2 \cap \ldots \cap A_n \) is in \( \alpha \).

We shall find several propositions concerning algebrae of sets useful. The first is the following:

**Proposition 1.5.1:** [1] Given any collection \( C \) of subsets of \( X \), there is a smallest algebra \( \alpha \) which contains \( C \); that is, there is an algebra \( \alpha \) containing \( C \) such that if \( \mathcal{B} \) is any algebra containing \( C \), then \( \mathcal{B} \) contains \( \alpha \).

**Proof:** let \( \mathcal{F} \) be the family of all algebra (of subsets of \( X \)) that contain \( C \). Let \( \alpha = \cap \{ \mathcal{B} : \mathcal{B} \in \mathcal{F} \} \). Then \( \alpha \) is a sub collection of \( \alpha \), since each \( \mathcal{B} \) in \( \mathcal{F} \) contains \( C \). Moreover, \( \alpha \) is an algebra , for if \( A \) and \( B \) are in \( \alpha \), then for each \( \mathcal{B} \in \mathcal{F} \), we have \( A \cup B \in \bigcap \{ \mathcal{B} : \mathcal{B} \in \mathcal{F} \} \). Similarly, we see that if \( A \in \alpha \), then \( \overline{A} \in \alpha \). From the definition of \( \alpha \), it follows that if \( \mathcal{B} \) is an algebra containing \( C \), then \( \mathcal{B} \supset \alpha \). The smallest algebra containing \( C \) is called the algebra generated by \( C \).

**Proposition 1.5.2:** [1]

Let \( \alpha \) be an algebra of subsets and \( \{ A_\lambda \} \) a sequence of sets in \( \alpha \). Then there is a sequence
\[
    \bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i
\]

**Proof:**

Since the position is trivial when \( \{ A_\lambda \} \) is finite, we assume \( \{ A_\lambda \} \) to be an infinite sequence. Set \( B_1 = A_1 \) and for each natural number \( n \), define
\[
    B_n = A_n \sim [A_1 \cup A_2 \cup \ldots \cup A_{n-1}] = A_n \cap \overline{A_1} \cap \ldots \cap \overline{A}_{n-1}
\]
Since the complements and intersections of sets in \( \alpha \) are in \( \alpha \), we have each \( B_n \in \alpha \). We also have \( B_n \subset A_n \). Let \( B_n \) and \( B_m \), be two such sets and suppose \( m < n \). Then \( B_m \subset A_m \) and so
\[ B_m \cap B_n \subset A_m \cap B_n = A_m \cap A_n \cap ... \cap \bar{A}_m \cap ... \]
\[ = (A_m \cap \bar{A}_m) \cap ... \cap \emptyset \cap ... = \emptyset \]

Since \( B_i \subset A_i \), we have
\[ \bigcup_{i=1}^{n} B_i \subset \bigcup_{i=1}^{n} A_i \]

Let \( x \in \bigcup_{i=1}^{n} A_i \). Then \( x \) must belong to at least one of the \( A_i \)'s. Let \( n \) be the smallest value of such \( i \) such that \( x \in A_i \). Then \( x \in B_n \) and so \( x \in \bigcup_{n=1}^{\infty} B_n \). Thus
\[ \bigcup_{n=1}^{\infty} B_n \supset \bigcup_{n=1}^{\infty} A_n \]

And we have
\[ \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \]

An algebra \( a \) of sets is called a \( \sigma \)-algebra, or a Borel filed, if every union of a countable collection of sets in \( a \) is again in \( a \). That is if \( (A_i) \) is a sequence of sets, then \( \bigcup_{i=1}^{\infty} A_i \) must again be in \( a \). From De Morgan's laws it follows that the intersection of a countable collection of sets in \( a \) is again in \( a \). A slight modification of the proof of proposition 1.4.1 gives us the following proposition.

**Proposition 1.5.3:** [1] Given any collection \( C \) of subsets of \( X \), there is a smallest \( \sigma \)-algebra containing \( C \), then \( a \subset B \).

The smallest \( \sigma \)-algebra containing \( C \) is called the \( \sigma \)-algebra generated by \( C \).

### 1.6 The axiom of choice and infinite direct product

An important axiom in set theory is the so-called axiom of choice. The axiom is the following:

**Axiom of Choice:** Let \( C \) be any collection of nonempty sets. Then there’s a function \( F \) defined on \( C \) which assigns to each set \( A \in C \) an element \( F(A) \) in \( A \).

The function \( F \) is called a choice function and its existence may be thought of as the result of choosing for each of the sets \( A \) in \( C \) an element in \( A \). There is of course no difficulty in doing this if there are only a finite number of sets in \( C \), but we need the axiom of choice in case the collection \( C \) is infinite. If the sets in \( C \) are disjoint, we may think of the axiom of choice as asserting the possibility of selecting a “parliament” consisting of one member from each of the sets in \( C \). Let \( C = \{X_\lambda\} \) be a collection of sets indexed by an index set \( \Lambda \). We take the direct product, if we may like to identify a relation on \( X \) with its graph and define \( a \) to be collection of all sets \( \{x_\lambda\} \) indexed by \( \Lambda \) and having the property that \( x_\lambda \in X_\lambda \). If \( \Lambda = \{1,2\} \), we have our earlier definition of the direct product \( X_1 \times X_2 \) of the two sets \( X_1 \) and \( X_2 \). If \( z = \{x_\lambda\} \) is an element of \( X_1 \times X_2 \), we call \( x_\lambda \) the \( \lambda \)-th co-ordinate of \( z \).

If one of the \( X_\lambda \) is empty, then \( x_\lambda \) is also empty. The axiom of choice is equivalent to the converse statement: if none of the \( X_\lambda \) are empty, then \( x_\lambda \) is not empty.

### 1.7 Countable and uncountable sets

One of the most profound ideas of modern mathematics is George Cantor’s theory of the infinite (George Cantor 1845-1918). Cantor’s insight was that infinite sets can be compared by size, just as finite sets can. For instance, we think of the number 2 as less than the number 3; so a set with two elements is “smaller” than a set with three elements. We would like to have a similar notion of comparison for infinite sets. In this
section we will present Cantor's ideas; we will also give precise definitions of the terms “finite” and “infinite”

**Definitions 1.7.1:** [Riez and Nagy (1999)] Let A and B be sets. We say that A and B have the same cardinality if there is a function $f$ from A to B which is both one to one and onto (that is $f$ is a bijection from A to B). We write $\text{card} (A) = \text{card} (B)$

**1.8 Countable sets**

We know that a set is countable if it was the range of some sequence. If it is the range of an infinite sequence may also be finite. In fact, every nonempty finite set is the range of an infinite sequence. For example, the finite set $\{x_1, ..., x_n\}$ is the range of the infinite sequence defined by setting $x_i = x_n$ for $i > n$. Thus, a set is countable infinite if it is the range of some infinite sequence but not the range of any finite sequence. The set $N$ of natural numbers is an example of a countable infinite set.

Before proceeding further we had better come to terms with the empty set. The empty set is not the range of any sequence (unless we admit sequence with zero terms). It is convenient, however to define finite and countable sets that the empty set is both finite and countable.

**Definition 1.8.1:** [8] A set is called finite if it is either empty or the range of a finite sequence. A set is called countable (or denumerable) if it is either empty or the range of a sequence. Thus, we may if we like identify a relation on X with its graph and define a missing word at once from this definition that the image of any countable set is countable, that is, that the range of any function with a countable domain is itself countable, and similarly for finite sets. We note that any set with that can be put in one to one correspondence with a finite set is finite and that any set can be put in one to one correspondence with a countable set must be countable. Since the set $N$ of natural numbers is countable but not finite, any set which can be put in one to one correspondence with $N$ is countably infinite. Thus, it shows that our definition is equivalent to the customary one, we must show that if an infinite set $E$ is the range of a sequence $(x_n)$, then $E$ can be put in one to one correspondence with $N$. To do this we define a function $\varphi$ from $N$ into $N$ by revision as follows: let $\varphi(1) = 1$, and define $\varphi(n + 1)$ to be the smallest value of $m$ such that $x_m \neq x_i$ for all $i \leq \varphi(n)$. Since $E$ is infinite, such an $m$ always exists, and by the well-ordering principle for $N$ there is always at least such $m$. The correspondence $n \rightarrow x_{\varphi(n)}$ is a one to one correspondence between $N$ and $E$. Thus we have shown that a set is countable infinite if and only if it can be put into one to one correspondence with $N$. We are now in apposition to prove some simple propositions about countable sets:

**Proposition 1.8.1:** [8] Every subset of a countable set is countable.

**Proof:** Let $E = \{x_n\}$ be a countable set, and let $A$ a subset of $E$. If $A$ is empty, $A$ is countable by definition. If $A$ is not empty, choose $x$ in $A$, define a new sequence $(y_n)$ by setting $y_n$ and is therefore countable.

**Proposition 1.8.2:** [8] Let $A$ be a countable set. Then the set of all finite sequences from $A$ is also countable.

**Proof:** since $A$ is countable, it can be put into one to one correspondence with a subset of the set $N$ of natural numbers. Thus it suffices to prove that the set $S$ of all finite sequence of natural numbers is countable. Let $\{2, 3, 5, 7, 11, ..., P_k, ...\}$ be the sequence of prime numbers. Then each $n$ in $N$ has a unique factorization of the form $n = 2^{x_1}3^{x_2}P_k^{x_k}$, where $x_i \in N_0 = N \cup \{0\}$ and $x_k > 0$. Let $f$ be the function on $N$ that assigns to the natural number $n$ the finite sequence $(x_1, ..., x_k)$ from $N_0$. Then $S$ is a subset of the range of $f$. Hence, is countable by proposition 1.7.1

**Proposition 1.8.3:** [8] the set of all rational numbers is countable.

**Proposition 1.8.4:** [8] The union of a countable collection of countable sets is countable.
Proof: Let \( C \) be a countable be a countable collection of countable sets. If all the sets in \( C \) are empty, the union is empty and thus countable. Thus we may as well assume that \( C \) contains nonempty sets, and since the empty set contributes nothing to the union of \( C \), we can assume that the sets in \( C \) are nonempty. Thus \( C \) is the range of an infinite sequence \( (A_n)_{n=1}^\infty \), and each \( A_n \) is the range of an infinite sequence \( (x_{nm})_{m=1}^\infty = 1 \). But the mapping of \( (n,m) \) to \( x_{nm} \) is a mapping of the set of ordered pairs of natural numbers onto the union of \( C \). Since the set of pairs of natural numbers is countable the union of the collection \( C \) must also be countable.

1.9 Relations and equivalence

Two given entities \( x \) and \( y \) may be “related” to each other in many ways as in \( x = y, x \in y, x \subset y, \) or for numbers \( x < y \). In general we say that \( R \) denotes a relation if, given \( x \) and \( y \), either \( x \) stands in the relation \( R \) to \( y \) (written \( x R y \)) or \( x \) does not stand in the relation \( R \) to \( y \). A relation \( R \) is said to be a relation on a set \( X \) if \( x R y \) implies \( x \in X \) and \( y \in X \). If \( R \) is a relation on a set \( X \), we define the graph of \( R \) to be the set \( \{(x,y): x R y \} \). Since we consider two relations \( R \) and \( S \) to be the same if \( \{x R y\} \leftrightarrow \{x S y\} \), each relation on a set \( X \) is uniquely determined by its graph, and conversely each subset of \( X \times X \) is the graph of some relation on \( X \). Thus we may if we like, identify a relation on \( X \) with its graph and define a solution to be a subset of \( X \times X \), in many formalized treatment of set theory a relation is in general defined simply as set of ordered pairs.

A relation \( R[3] \) is said to be transitive on a set \( X \) if \( x R y \) and \( x R z \) for all \( x, y, \) and \( z \in X \). Thus \( = \) and \( < \) are transitive relations on a set of real numbers. A relation \( R \) is said to be symmetric on \( X \) if \( x R y \) implies \( y R x \) for all \( x, y \) in \( X \). It is said to be reflexive on \( X \) if for all \( x \in X \) we have \( x R x \).

A relation that is transitive, reflexive and symmetric on \( X \) is said to be an equivalence relation on \( X \) or simply equivalence on \( X \).

Suppose that \( \equiv \) is an equivalence relation on a set \( X \), for a given \( x \in X \), let \( E_x \) be the set of elements equivalent to \( x \), that is \( E_x = \{ y: y \equiv x \} \). If \( y \) and \( z \) are both in \( E_x \), then \( y \equiv x \) and \( z \equiv x \) and by symmetry and transitivity we have \( z \equiv y \), thus any two elements of \( E_x \) are equivalent. If \( y \in E_x \) and \( z \equiv y \), then \( z \equiv y \) and \( y \equiv x \), whence \( z \equiv x \) and so \( z \in E_x \). Thus any element of \( E_x \) equivalent to an element of \( E_x \).

Consequently, for any two elements \( x \) and \( y \) of \( X \), the sets \( E_x \) and \( E_y \) are either identical (if \( x \equiv y \)) or disjoint (if \( x \not\equiv y \)). The sets in the collection \( \{E_x: x \in X \} \) are called equivalence sets or classes of \( X \) under \( \equiv \).

Note that \( x \in E_x \) and so no equivalent class is empty. The collection of equivalence classes under an equivalence \( \equiv \) is called the quotient of \( X \) with respect to \( \equiv \) and is sometimes denoted by \( X/\equiv \). The mapping \( x \rightarrow E_x \) is called the natural mapping of \( X \) onto \( X/\equiv \).

A binary operation on \( X \) is mappings from \( X \times X \) to \( X \). We say that an equivalent relation \( \equiv \) is compatible with a binary operation \( + \) if \( X \equiv X^1 \) and \( Y \equiv Y^1 \) imply that \( (x+y) \equiv (x_1+y_1) \). In this case \( + \) defines an operation on the quotient \( Q \equiv X/\equiv \) as follows: if \( E \) and \( F \) belong to \( Q \), choose \( x \in E, y \in F \) and 

1.10 Partial ordering and maximal principle

A relation \( R[3] \) is said to be a missing word on a set \( X \) if \( x R y \) and \( y R x \) imply \( x = y \) for all \( x \) and \( y \) in \( X \). A relation \( < \) is said to be a partial ordering of a set \( X \) (or to partial order \( X \)) if it is transitive and antisymmetric on \( X \). Thus \( \leq \) is a partial ordering on real numbers and \( \subset \) is a partial ordering on \( p(X) \). A partial ordering \( < \) on a set of two elements \( x \) and \( y \) of \( X \) we have either \( x < y \) and \( y < x \) thus \( \leq \) linearly orders the set real numbers, while \( \subset \) is not a linear ordering on \( p(X) \).

If \( \prec \) is a partial order on \( X \) and if \( a \prec b \), we often say that \( a \) precedes \( b \) or that \( b \) follows \( a \). Sometime we say that \( a \) is less than \( b \) or \( b \) is greater than \( a \). If \( E \subset X \), we say that an element \( a \in E \) is the first element in
E or the smallest element in E if whenever and \( x \neq a \), we have \( a < x \). similarly for last (or largest) elements. An element \( a \in E \) is called a minimal element of E if there is no \( x \in E \) with \( x \neq a \) and \( x < a \), and similarly for maximal elements. It should be observed that if a set has a smallest element, then that element is a minimal element. If \( < \) is a linear ordering, a minimal element is a least element, but in general it is possible to have minimal elements which are not least elements.

Our definition of partial order makes no assertion about the possibility or necessity of \( x < x \). if we have \( x < x \) for all \( x \), we call \( < \) a reflexive partial order. If we never have \( x < x \) then \( < \) is called a strict partial order. Thus \( < \) is a strict partial order for the real numbers and \( \leq \) is a reflexive partial order. To any partial order \( < \) there is association a unique strict partial order and a unique reflexive partial order that agree with \( < \) for all \( (x, y) \) with \( x \neq y \). If \( < \) is any partial order, we use \( \leq \) for the associated reflexive partial order. The following principle is equivalent to the axiom of choice and is often more convenient to apply.

**Hausdorff Maximal Principle:** let \( < \) be a partial ordering on a set \( X \). Then [2] there is a maximal linearly ordered subset \( S \) of \( X \), that is a subset \( S \) of \( X \) which is linearly ordered by \( < \) and has the property that if \( S \subset T \subset X \) and \( T \) is linearly ordered by \( < \), then \( S = T \).

### 1.11 Well ordering and the countable ordinals

As strict linear ordering \( < \) on a set \( X \) is called a well ordering for \( X \) or is said to well order \( X \) if every nonempty subset of \( X \) contains a first element. Thus, if we take \( X = N \) and \( < \) to mean less than, then \( N \) is well ordered by \( < \). On the other hand, the set \( R \) of all real numbers is not well ordered by the relation “less than”. The following principle clearly implies the axiom of choice and can be shown equivalent to it.

**Well Ordering Principle:** Every set \( X \) can be well ordered; that is, there is a relation \( < \) that well orders \( X \).

**Proposition 1.11.1:** (Riez and Nagy (2002)). There is an uncountable set \( X \) that is well ordered by a relation \( < \) in such a way that exist a last element \( \Omega \) in \( X \). If \( x \in X \) and \( x \neq \Omega \), then the set \( \{ y \in X : y < x \} \) is countable.

**Proof:** let \( Y \) be any uncountable set, by the well ordered principle, there is a well ordering \( < \) for \( Y \). if \( Y \) does not have a last element, taking an element \( a \not\in Y \), replace \( Y \) by \( Y \cup \{ a \} \) and extend the order \( < \) by setting \( y < a \) for all \( y \in Y \). this new \( Y \) has a last element and is well ordered by \( < \). The set of \( y \) in \( Y \) for which the set \( \{ x \in Y : x < y \} \) is uncountable is a nonempty set, since it contains the last element of \( Y \). Let \( \Omega \) be the smallest element in this set and let \( X = \{ x \in Y : x < \Omega \text{ or } x = \Omega \} \). The \( X \) is the required set.

The well-ordered set \( X \) given in the proposition will be very useful for constructing examples. It can be shown to be unique in the sense that if \( Y \) is any other well-ordered set with the same properties, then there is a one to one order preserving correspondence between \( X \) and \( Y \). The last element \( \Omega \) in \( X \) is called the first uncountable ordinal and \( X \) is called the set of ordinals less than or equal to the first uncountable ordinal. The elements \( x < \Omega \) are called countable ordinals. If \( y : y < \) is finite, we call \( x \) a finite ordinal. If \( \omega \) is the first nonfinite ordinal, then \( \{ x : x < \omega \} \) is the set of finite ordinals and is equivalent as an ordered set, to set \( N \) of positive integers.

### 2 Set in Elements of Real Analysis

#### 2.1 Introduction axioms of the real numbers

The present section is devoted to a review and systematization of those results which will be useful later.

One approach to the subject of real numbers is to define them as Dedekind cuts of rational numbers, the rational numbers in turn being defined in terms of the natural numbers. Such a program gives an elegant or
construction of the real numbers out of more primitive concepts and set theory. We shall not concern ourselves here with the construction of the real numbers but will think of them as already given, and list a set of axioms for them.

We thus assume as given the set \( \mathbb{R} \) of real numbers, the set \( \mathbb{P} \) of positive real numbers, and the functions \( + \) and \( \cdot \) on \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{R} \) and assume that these satisfy the following axioms, which we list in three groups. The first group describes the algebraic properties. The third comprises the least upper bound axiom.

**Theorem 2.1** [9] **The Field Axioms:** for all real number \( x, y \) and \( z \) we have

A1. \( x + y = y + x \)
A2. \( (x + y) + z = x + (y + z) \)
A3. \( \exists 0 \in \mathbb{R} \) such that \( x + 0 = x \) for all \( x \in \mathbb{R} \)
A4. For each \( x \in \mathbb{R} \) there is \( a \in \mathbb{R} \) such that \( x + a = 0 \)
A5. \( ax = xa \)
A6. \( (xy)z = x(yz) \)
A7. \( \exists 1 \in \mathbb{R} \) such that \( 1 \neq 0 \) and \( x \) for all \( x \in \mathbb{R} \).
A8. For each \( x \) in \( \mathbb{R} \) different from 0 there is \( w \in \mathbb{R} \) such that \( xw = 1 \)
A9. \( x(y + z) = xy + xz \).

Any set that satisfies these axioms is called a field (under + and \( \cdot \)). The second classes of properties possessed by the real numbers have to do with the fact that numbers are ordered.

**Theorem 2.2 Axiom of Order** [9]: The subset \( \mathbb{P} \) of positive real numbers satisfies the following:

B1. \( (x, y \in \mathbb{P}) \rightarrow x + y \in \mathbb{P} \).
B2. \( x, y \in \mathbb{P} \rightarrow xy \in \mathbb{P} \)
B3. \( (x \in \mathbb{P}) \rightarrow -x \in \mathbb{P} \)
B4. \( (x \in \mathbb{R}) \rightarrow (x = 0) \) or \( (-x \in \mathbb{P}) \)

Any system satisfying the axioms of groups A and B is called an ordered field. Thus the real numbers are an ordered field. The rational numbers give another example of an ordered field.

In an ordered field we define the notion \( x < y \) to mean \( y - x \in \mathbb{P} \). We write \( x \leq y \) or \( x = y \) for \( x < y \) or \( x = y \). In terms of \( < \) axiom B1 is equivalent to

\[(x < y \& z < w) \rightarrow x + z < y + w\]

B2 is equivalent to

\[(0 < x < y \& 0 < z < w) \rightarrow xz < yw\]

Axiom B3 asserts that a number cannot be both greater than and less another, while B4 states that of any two different numbers one must be larger. Since axiom B1 implies that the relation \( < \) is transitive, we see that the real numbers are linearly ordered by \( < \). Except for the discussion in the beginning of the next section we take all the consequences of these two axiom groups for granted and use them without explicit mention.

The third axiom group consists of a single axiom and it is this axiom that distinguishes the real numbers from other ordered fields.

**Theorem 2.3 Completeness Axiom** [10]: Every nonempty set \( S \) of real numbers which has an upper bound has a last upper bound.

As a consequence of axiom C we have the following preposition:
Proposition 2.1 [10]: Let \( L \) and \( U \) be nonempty subsets of \( R \) with \( R = L \cup U \) and such that for each \( l \) in \( L \) and each \( u \) in \( U \), we have \( l < u \). Then either \( L \) has a greatest element or \( U \) has a least element.

We shall often denote the least upper bound of \( S \) by \( \text{sup} \ S \) or by \( \text{sup} x \) and occasionally by \( \text{sup} \{x : x \in S\} \).

We can define lower bounds and greatest lower bounds in a greatest lower bound. We denote the greatest lower bound of a set \( S \) by \( \text{inf} S \) or by \( \text{inf} x \). Note that \( \inf x = - \sup -x, x \in S \).

Proposition 2.2 [10]: For any set \( E \) with \( E \) closed; that is \( E = E^c \).

Proposition 2.3: The union \( F_1 \cup F_2 \) of two closed sets \( F_1 \) and \( F_2 \) is closed.

2.2 The natural and rational numbers as subsets of \( R \)

By the principle of recursive definition [3] there is a function \( \varphi \) from the natural numbers to the real numbers defined by \( \varphi(1) = 1 \) and \( \varphi(n + 1) = \varphi(n) + 1 \). (here \( 1 \) denotes a real number on the right side and a natural number on the left). We shall show that the mapping \( \varphi \) is a one to one mapping of \( N \) into \( R \).

Let \( p \) and \( q \) be two different natural numbers, say \( p < q \), then \( q = p + n \), and we shall show that \( \varphi(p) < \varphi(q) \) by induction on \( n \). For \( n = 1 \) we have \( \varphi(p + n + 1) = \varphi(p) + 1 > \varphi(p) \). For general \( n \) we have \( \varphi(p + n + 1) = \varphi(p) + n \), and so \( \varphi(p + n) > \varphi(p) \) implies \( \varphi(p + n + 1) > \varphi(p) \). Thus by induction \( \varphi(p + n) > \varphi(p) \) and we see that the mapping \( \varphi \) is one to one. We can also prove by induction that \( \varphi(p + q) = \varphi(p) + \varphi(q) \) and \( \varphi(p \cdot q) \). Thus \( \varphi \) gives a one to one correspondence between the natural numbers and a subsets of \( R \) and \( \varphi \) preserves sums, products and the relation \( < \). Strictly speaking, we should distinguish between the natural number \( n \) and its image \( \varphi(n) \) and \( \varphi \), but we shall not make the distinction here; we shall consider the set \( N \) of natural numbers to be a subset of \( R \). By taking differences of natural numbers, we obtain the integers as subsets of \( R \) and taking quotients of integers gives us the rational. Since axiom \( C \) was not used in this discussion, the same results hold for any ordered field. Thus we have shown the following:

Proposition 2.4 [4]: Every ordered field contains (sets isomorphic to) the natural numbers, the integers and the rational numbers.

The set of extended real numbers, we extend the definition of \( < \) to the extended real numbers by postulating \( -\infty < x < \infty \), for each real number \( x \), we define

Theorem 2.4 [5] Axiom of Archimedes: Given any real number \( x \), there is an integer \( n \) such that \( x < n \).

Proof: If \( x < 0 \), take \( n = 0 \). Otherwise the set of integers \( k \) such that \( k \leq x \) is nonempty. Since \( S \) has the upper bound \( x \), it has a least upper bound \( y \) by axiom \( C \). Since \( y \) is the least upper bound for \( Sy - \frac{1}{2} \) cannot be an upper bound for \( S \) and so there is a \( k \in S \) such that \( k > y - \frac{1}{2} \). But \( k + 1 > y + \frac{1}{2} > y \), and so \( (k + 1) \in S \). Since \( k + 1 \) is an integer not in \( S \), we must have \( k + 1 \) greater than \( x \) by the definition of \( S \).

Corollary 2.1: Between any two numbers is a rational; that is if \( x < y \), then there is a rational \( r \) with \( x < r < y \).

2.3 The extended real numbers

It is often convenient [4] to extend the system of real numbers by the addition of two elements, \( +\infty \) and \( -\infty \). This enlarge set is called the set of extended real numbers. We extend the definition of \( < \) to the extended real numbers by postulating \( -\infty < x < \infty \), for each real number \( x \), we define

\[
x + \infty = \infty, \quad x - \infty = -\infty
\]
Proposition 2.5.1

\[ x. \infty = \infty \quad \text{if } x > 0 \]
\[ x. -\infty = -\infty \quad \text{if } x > 0 \]

For all real numbers \( x \), and set
\[ \infty + \infty = \infty, \quad -\infty - \infty = -\infty \]
\[ \infty, (\pm \infty) = \pm \infty, \quad -\infty, (\pm \infty) = \pm \infty \]

The operation \( \infty - \infty \) is left undefined, but we shall adopt the arbitrary convention that \( \infty = 0 \).

A function whose values are in the set of extended real numbers is called an extended real valued function.

2.4 Sequences of real numbers

By a sequence \((x_n)\) of real numbers we [7] mean a function that maps each natural number \(n\) into the real number \(x_n\). We say that real number \(L\) is a limit of the sequence \((x_n)\) if for each positive \(\varepsilon\) there is an \(N\) such that for all \(n \geq N\) we have \(|x_n - l| < \varepsilon\) it is easily verified that a sequence can have at most one limit, and we denote this limit \(x_n\) when it exists. In symbols \(L = \lim x_n\) if
\[ (\varepsilon > 0)(\exists N)(n \geq N)(|x_n - l| < \varepsilon). \]

A sequence \((x_n)\) of real numbers is called a Cauchy sequence if given \(\varepsilon > 0\), there is an \(N\) such that for all \(n \geq N\) we have \(|x_n - x_m| < \varepsilon\). The Cauchy Criterion states that a sequence of real numbers converges if and only if it is a Cauchy Sequence.

If \(l = x_n\) we often write \(x_n \to l\), if in addition, \((x_n)\) is monotone, that is \(x_n \leq x_{n+1}\), we write \(x_n \uparrow l\). If \((x_n)\) is a sequence, we define its limit superior by
\[ \lim_{\inf} x_n = \sup_{\inf} x_k \]
We have \(\lim (-x_n) = -\lim x_n\) and \(\lim x_n \leq \lim_{\inf} x_\cdot\). The sequence \((x_n)\) converges to an extended real number \(l\) if and only if \(l = \lim_{\inf} x_n = \lim x_n\). If \((x_n)\) and \((y_n)\) are two sequences, we have
\[ \lim x_n + \lim y_n \leq \lim (x_n + y_n) \leq \lim_{\inf} x_n + \lim_{\inf} y_n \]
Provided no sum is of the form \(\infty - \infty\), \(|x - y| < \varepsilon\) belong to \(O\) and hence to \(U\), since \(O < U\).

2.5 Open and closed sets of real numbers

The simplest sets of real numbers are the intervals. We define the open interval \((a, b)\) to be the set \(\{x: a < x < b\}\). We always take \(a < b\), but we consider also the intervals \((-\infty, a) = \{x: a < x\}\) and \((a, \infty) = \{x: x < b\}\). Sometimes we write \((-\infty, \infty)\) for the set of real numbers. We define the closed interval of the set \(\{x: a \leq x \leq b\}\), for closed intervals we have \(a\), and it always assume that \(a < b\). The half-open interval \([a, b)\) is closed to \(x: a < x \leq b\), and \([a, b) = \{x: a \leq x \leq b\}\). A generalization of the notion of an open interval is given by that of a closed set:

Definition 2.5.1 [10]: A set \(O\) of real numbers is called open if for each \(x \in O\) there is a \(\delta > 0\) such that each \(y\) with \(|x - y| < \delta\) belongs to \(O\).

Proposition 2.5.1 [9]: The intersection \(O_1 \cap O_2\) of two open sets \(O_1\) and \(O_2\) is open.
Corollary 2.5.1: The intersection of any finite collection of open set is open.

Proposition 2.5.2 [3]: The union of any collection $O$ of open sets is open.

Proposition 2.5.3 [3]: Every open set of real number 5 is the union of a countable collection of disjoint open intervals.

Proposition 2.5.4 [8] (Lindelof): Let $O$ be a collection of open sets of real numbers. Then there is a countable sub collection $\{O_i\}$ of $O$ such that

$$\bigcup_{n \geq 0} O = \bigcup_{i=1}^{\infty} O_i$$

We shall also study the notion of a closed set, which generalizes the notion of a closed interval. We begin by defining a point of closure.

Definition 2.5.2 [8]: A real number $x$ is called a point of closure of a set $E$ if for every $\delta > 0$ there is a $y \in E$ such that $|x - y| < \delta$.

This is equivalent to saying that $x$ is a point of closure of $E$ if every open interval containing $x$ also contains a point of $E$. Each point of $E$ if every open interval containing $x$ also contains a point of $E$. Each point of $E$ is trivially a point of closure of $E$, we denote the set of points of closure of $E$ by $E^-$. Thus $E \subset E^-$.

Proposition 2.5.5 [2]: A set $F$ is called closed if $F = F^-$

Proposition 2.5.6 [2]: For any set $E$ the $E$ is closed; that is $E = E^-$

Proposition 2.5.7 [2]: The union $F_1 \cup F_2$ of two closed sets $F_1$ and $F_2$ is closed.

Proposition 2.5.8 [2]: The intersection of any collection $O$ of closed sets is closed

Proposition 2.5.9 [2]: The complement of an open set is closed and the complement of a closed set is open.

Proof: Let $O$ be open, if $x \in O$, there is a $\delta > 0$ such that if $|x - y| < \delta$, then $y \in O$. Hence $x$ cannot be point of closure of $O$, since there is no $y \in O$ with $|x - y| < \delta$. Thus $O$ contains all of its points of closure and is therefore closed.

On the other hand, let $F$ be closed and $x \in F$. Then, since $x$ is not a point of closure of $F$, there is a $\delta > 0$. Such that there is no $y \in F$ with $|x - y| < \delta$, then $y \not\in F$. Thus $F$ is open. That is if $O$ is a collection of open sets such that $F \subset \bigcup \{O: O \in O\}$. Then there is a finite collection $\{O_1, ..., O_n\}$ of sets in $O$ such that

$$F \subset \bigcup_{i=1}^{n} O_i$$

Proposition 2.5.10 [8]: Let $O$ be a collection of closed sets (of real numbers) with the property that every finite subcollection of $O$ has a nonempty intersection and suppose that of the sets in $O$ is bounded then

$$\cap F \neq \emptyset$$

Theorem (Heine-Borel) [3]: Let $F$ be closed and bounded set of real numbers. Then each open covering of $F$ has a finite sub covering $I = (f(x) - \varepsilon, f(x) + \varepsilon)$ is an open set and so its inverse image $f^{-1}[I]$ must be open, since $x \in f^{-1}[I]$, there must be some $\delta > 0$ such that $(x - \delta, x + \delta) \subset f^{-1}[I]$.

Proposition 2.5.11 (Intermediate Value Theorem) [3]: Let $f$ be continuous real valued function on $[a, b]$ and suppose that $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$; then there is a point $c \in [a, b]$ such that $f \left(\frac{c}{n}\right) = y$. 

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**Definition 2.5.3 [10]:** A real valued $f$ defined on a set $E$ is said to be uniformly continuous on $E$ if given $\varepsilon > 0$, there is a $\delta > 0$. Such that for all $x$ and $y$ in $E$ with $|x − y| < \delta$ we have $|f(x) − f(y)| < \varepsilon$.

**Proposition 2.5.12 [10]:** If a real valued function $f$ is defined and continuous on a closed and bounded set $F$ of real numbers, it is uniformly continuous on $F$.

**Proposition 2.5.13 [9]:** Let $f$ be a real valued function defined and continuous on a closed and bounded set $F$. Then $f$ is bounded on $F$ and assumes its maximum and minimum on $F$: that is there are points $x_1$ and $x_2$ in $F$ such that $f(x_1) \leq f(x_2)$ for all $x$ in $F$.

**Proposition 2.5.14 [9]:** Let $f$ be a real valued function defined on $(-\infty, \infty)$. Then $f$ is continuous if and only if for each open $O$ of real numbers $f^{-1}[O]$ is an open set.

**Definition 2.5.4 [4]:** A sequence $(f_n)$ of functions defined on a set $E$ is said to converge pointwise on $E$ to a function $f$ if for every $x$ in $E$ we have $f(x) = \lim f_n(x)$ that is if given $x, \varepsilon > 0$, there is an $N$ such that for all $n \geq N$, we have $|f(x) − f_n(x)| < \varepsilon$.

### 2.6 Borel sets

Although the intersection of any collection of closed sets is closed and the union of any finite collection of closed sets is closed. The unions of a countable collection of closed sets need not to be closed. For example, the set of rational numbers is the union of borel sets is the smallest $\sigma$-algebra which contains all of the open sets. A set which is countable union of closed sets is called an $F(F$ for closed, $\sigma$ for sum). Thus, every countable set is an $F_\sigma$ as of course, every closed set. A countable union of sets in $F_\sigma$ is again in $F_\sigma$ since

$$(a, b) = \bigcup_{n=1}^{\infty} \left[ a + \frac{1}{n}, b - \frac{1}{n} \right].$$

Each open interval is an $F_\sigma$ and hence each open set is an $F_\sigma$, we say that a set is a $G_\delta$ if it is the intersection of a countable collection of open set ($G$ for open, $\delta$ for durchschnitt). Thus, the complement of an $F_\sigma$ is a $G_\delta$ and conversely, the $F_\sigma$ and $G_\delta$ are relatively simple types of Borel sets. We could also consider sets of type $F_\sigma$, which are the intersections of countable collections of sets each of which is an $F_\sigma$. Similarly, we can construct the classes $G_{\sigma\sigma}, F_{\sigma\sigma}$ etc. thus the classes in the two sequence.

$F_\sigma, F_{\sigma\sigma}, F_{\sigma\sigma\sigma}, \ldots, G_{\sigma\sigma\sigma}, G_{\sigma\sigma\sigma\sigma}, \ldots$

are all classes of Borel sets. However, not every Borel set belong to one of these classes, but we shall need only the properties that follow directly from the fact they form the smallest $\sigma$-algebra containing the open and closed sets.

### 3 Set in Elements of Metric Space Topology

#### 3.1 Open balls; closed balls; spheres

In this Section, we consider some important types of sets in metric spaces; these are called open balls, closed balls and spheres. The sets will play crucial roles in the rest of our study of metric spaces.
**Definition 3.1** [10]: let \((X, \rho)\) be a metric space. Given a point and \(x\) such that \(x > 0\),

(i) \(B_r(x_0) = \{y \in X: \rho(x_0, y) < r\}\) is called an open ball centered at \(x_0\) of radius \(r\). Similarly, for a point \(x_0 \in X\) and a positive real number \(r\)

(ii) \(B_r(x_0) = \{y \in X: \rho(x_0, y) \leq r\}\) is called the closed ball centered at \(x_0\) of radius \(r\), finally, with \(x\) and \(x_0\) as above the set

(iii) \(S_r(x_0) = y \in X: \rho(y, x_0) = r\) will be called the sphere of radius \(r\) centered at \(x_0\)

**Example 3.1** [10]: Let \(X \in R\) (the real line) with metric \(\rho_0\) defined by

\[
\rho_0(x, y) = \begin{cases} 
1, & x \neq y \\
0, & x = y
\end{cases}
\]

for arbitrary \(x, y \in R\). Describe the open balls:

(i) \(B_{\frac{1}{2}}(1)\);
(ii) \(B_1(1)\);
(iii) \(B_2(1)\);
(iv) \(B_1(5)\);
(v) \(B_\frac{1}{2}(4)\)

**Solution**

(i) \(B_0(1) = \{y \in R: \rho_0(y, 1) < 1/2\}\) (definition). But \(\rho_0(x, y)\) has only two values for all \(x, y \in R\), i.e. 0 or 1. Thus, in particular, \(\rho_0(y, 1) < 1/2\) implies that it cannot be equal to 1. Hence, \(\rho_0(y, 1)\) must be equal to 0. Thus \(B_\frac{1}{2}(1) = \{y \in R: \rho_0(y, 1) = 0\}\). Since \(\rho_0\) is a metric, it follows that \(\rho_0(x, y), 0 > xy\). In particular, \(\rho_0(y, 1) = 0 > y - 1\). Hence, \(B_{\frac{1}{2}}(1) = \{y \in R: \rho_0(y, 1) = 0\}\).

Hence the open ball \(B_\frac{1}{2}(1) = \{1\}\), the singleton \(\{1\}\).

(ii) \(B_1(1) = \{y \in R: \rho_0(y, 1) < 1\}\), but \(\rho_0(y, 1) = 0\) or 1. Since \(\rho_0(y, 1) = 1\), it cannot be equal to 1, hence, \(\rho_0(y, 1) = 0\), so that \(B_1(1) = \{y \in R: \rho_0(y, 1) < 1\}\).\(\{y \in R: \rho_0(y, 1) = 0\}\). Again, since \(\rho\) is a metric, we have that for each pair \(x, y \in R, \rho_0(x, y) = 0 > x + y\). In particular, \(\rho_0(y, 1) = 0, y = 1\), hence, \(B_1(1) = \{1\}\), a singleton.

(iii) \(B_2(1) = \{y \in R: \rho_0(y, 1) < 2\}\). But \(\rho_0(x, y) = 0\) or 1, and so in any case \(n\) is less than 2 for all \(x, y \in R\). Hence, \(B_{\frac{1}{2}}(1) = \{y \in R: \rho_0(y, 1) < 2\}\). Thus, \(B_1 - R\) (the whole space, \(R\)).

\[
B_{\frac{1}{2}}(2) = \{y \in R: \rho_0(y, 2) < \frac{1}{2}\} \{y \in R: \rho_0(y, 2) \geq \frac{1}{2}\}
\]

(iv) \(B_1(5) = \{y \in R: \rho_0(y, 5) = 1\}\). But \(\rho_0(y, 5) = 0\) or 1 so that \(\rho_0(y, 5) = 1 > \rho_0(y, 5) = 0\). Thus, \(B_1(5) = \{y \in R: \rho_0(y, 5) = 1\}\).\(\{y \in R: \rho_0(y, 5) = 0\}\) i.e. \(B_1(5) = \{5\}\), singleton.

(v) \(B_{\frac{1}{2}}(1) = \{y \in R: \rho_0(y, 1) < \frac{3}{2}\}\). But \(\rho_0(x, y) = 0\) or 1 for all \(x, y \in R\). In either case, \(\rho_0(x, y)\) is always less than 3/2 for all \(x, y \in R\). Hence, \(B_{\frac{1}{2}}(1) = \{y \in R: \rho_0(y, 1) < \frac{3}{2}\}\).\(\{y \in R\}\) = \(R\), i.e. \(B_1(1) = R\) (the whole space).

**Example 3.2** [10]: Let the set \(R^2\) of all ordered pairs of real numbers with metric \(S_2\) defined by

\[
S_2(\bar{x}, \bar{y}) = \sqrt{\sum_{i=1}^{2} (x_i - y_i)^2}
\]

where \(\bar{x} = (x_1, x_2)\) and \(\bar{y} = (y_1, y_2)\) be a metric space.

Describe the sets:

(i) \(B_1(0)\)
(ii) \(B_{\frac{1}{2}}(0)\)
(iii) \(S_2(0)\), where \(0 = (0, 0)\) in \(R^2\)
(iv) \(B_r(\bar{x})\) for arbitrary \(x_0 \in R^2\) and \(r > 0\)
Fig. 1. Closed ball

Solution

\[ B_1(0) = \{ u = (x, y) \in \mathbb{R}^2 : \rho_2(u, 0) < 1 \} \]

\[ = \{ u = (x, y) \in \mathbb{R}^2 : \rho_2(x, y), (0, 0) \{ < 1 \} \}

\[ = \left\{ u \in \mathbb{R}^2 : \left[ (x - 0)^2 + (y - 0)^2 \right]^{\frac{1}{2}} < 1 \right\} \text{ definition of } \rho_2 \]

\[ = \{ u = (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \} \]

= interior of the unit circle centered at the origin see Fig. 1.

(ii) with identical computations as in (i) above but with "<" replaced by "\leq" we obtain

\[ B_1(0) = \{ u = (x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \} \]

Which is the unit disc centered at the origin, (Fig. 2.)

Fig. 2. Closed ball
(iii) $S_1(0)$ is computed as in (i) above but with " < " replaced by " = ". Hence we obtain the unit circle centered at the origin (see Fig. 3.)

![Fig. 3. Open ball](image)

(iv) $B_r(x_0) = \{u = (x,y) \in \mathbb{R}^2: \rho_2 = |(x,y) + (x_0, y_0)| < r \}$

\[= \{u \in \mathbb{R}^2: |(x,x_0)^2 + (y-y_0)^2|^{1/2} < r \}\]

\[= \{u \in \mathbb{R}^2: (x - x_0)^2(y - y_0)^2 < r^2 \}\]

Hence, $B_r(x_0)$ is the inside of circle centered at $X_0 = (x_0, y_0)$ with radius $r > 0$.

### 3.2 Open sets, closed sets

#### 3.2.1 Open sets

The most basic notion for a metric space is that of an open set. We discuss this next.

**Definition 3.2.1:** Let $(x, \rho)$ be a metric space and let $A$ be an arbitrary subset of $X$. The subset $A$ is said to be an open set in $X$ or is called open in $X$ if for each point $x \in A$, there exists a positive real number (ie. (0)) such that $B_r(x) \subset A$. Pictorially, we have the following

![Fig. 4. Closed set](image)
Theorem 3.2.1 [5]: in any metric space \((X, \rho)\) each open ball is an open set in \(R\)

**Proof:** Let \(B_1(x_0)\) be an arbitrary open ball in \((X, \rho)\). Let \(B \cap B_0 \in X\). We want to prove that \(B\) is an open set in \(X\), this means that for any arbitrary \(x \in B\), we can find some \(r > 0\) such that \(x \in B_1\), so let \(x \in B_1\) and \(y \in B_0 \cap B_1 - B_0\) be arbitrary. This implies \(x \in B_1\) for each \(i\). But each \(B_1\) is an open set in \(X\) so there exists some \(\alpha\) such that \(x \in \bigcap B_1(x)\) is contained in each \(B_1\). We have \(x \in B_1(x) \cap B_1 - B_1\), so \(B \cap B_1\) is an open set in \(X\).

Example 3.2.1 [10]: Let \(X = R\) (the real) with the usual metric and using Fig. 5, let

\[
U_1(0) = \{x: -1 < x < 1\} \\
U_2(0) = \{x: \frac{1}{2} < x < \frac{1}{2}\} \\
\vdots \\
U_n(0) = \{x: -\frac{1}{n} < x < \frac{1}{n}\}
\]

Then clearly each \(U_n(0)\) is an open bounded interval and hence is an open set in \(R\) (with the usual metric) but the intersection \(\bigcap_{n=1}^{m} U_n(0)\) and the singleton set \([0]\) in \(R\) (with the usual metric) is not open in \(R\). Hence, \(\bigcap U_n(0)\) is not an open set in \(R\).

Theorem 3.2.2 [4]: Let \((X, \rho)\) be a metric space. A subset \(A\) of \(X\) is an open set in \(R\) if and only if \(A\) is a union of open balls.

**Proof:**

Let \(A\) be an open set in \((X, \rho)\), we want to show that \(A\) is a union of open balls. Let \(X \in A\), since \(A\) is an open set in \(X\), there exists some real numbers \(r > 0\) such that \(B_r(x) \subset A\). In the same manner, it follows that for each \(y \in A\), we can find \(r_y > 0\) (a real number depending on the particular \(y\)) such that \(B_{r_y}(y) \subset A\), then clearly, \(A\) is a union of open balls.
Fig. 6. Closed union and intersection of balls

Let $A$ be expressible as the union of open balls in $X$ i.e. $A = \bigcup B_r(x)$, where $B_r(x)$ denotes an open ball centered at $x$ with radius $r$ and $B_r(x) \in X$. We want to prove that $A$ is an open set in $X$.

**Method 1:** To show that $A$ is an open set in $X$ we have to show that for arbitrary $x \subseteq A$ we can find some $r_x > 0$ such that $B_r(x) \subseteq A, x \in A$ be arbitrary. Since $A = \bigcup B_r(y), x$ must belong to at least one of the open balls $B_r(y), y \in A$, without loss of generality, we may say that $x \subseteq B_r(y)$, but $B_r(y)$ is open. Hence, there exists $r, x = 0, ...$

**Method 2:** $\bigcup B_r(x)$ each $B_r(x)$ is an open set contained in every open ball an open set. By theorem 3.2, $A = \bigcup B_r(x)$ is an open set in $X$.

We conclude this section with the following definition: let $(X, \rho)$ be a space and let $x \in X$. A subset $N_x$ of $X$ is said to be a neighborhood $x$ if $x \in N_x$ and if in $N$ there exists an open ball $B_r(x)$ for some $r > 0$ at $x$ such that $B_r(x) \subseteq N_x$. See Fig. 7.

Some authors call an open set a neighborhood

Fig. 7. Interior points
3.3 Interior point of a set

Definition 3.3.1 [7]: the set of all interior points of $A$ is called the interior of $A$ and is denoted by $A^0$ or $\text{int}(A)$. Consider the four intervals, $[0,1], (0,1], [0,1)$ and $(0,1)$ whose end point are 0 and 1. Clearly, with the usual metric, the interior (set of interior points of each is open interval $(0,1)$).

We conclude this section with the following theorem.

Theorem 3.3.1 [7]: A subset $A$ of a metric space $(X, \rho)$ is open if it is equal to its interior i.e. if and only if $A = A^0$

3.4 Limit point

Definition 3.4.1 [1]: Let $(X, \rho)$ be a metric space and let $A$, be a subset of $X$. The point $x \subset X$ (not necessarily in $A$). Is said to be an accumulation point of $A$ (or a limit point of $A$ or a cluster point of $A$, or a derive point of $A$) if every open ball centered at $x$ contains at least one point different from $x$

$$\cap (B_r(x) - x) \cap A \neq \emptyset, \quad r > 0$$

Remark: From the above definition, limit of a set $A$ and limit of a point are the same.

Definition 3.4.2 [1]: The set of limit points of $A$, denoted by $A'$, is called the derived set of $A$.

Definition 3.4.3 [1]: A subset of a metric space $(X, \rho)$ is closed if and only if it contains all its limit points. With this in mind to show that $A$ is closed, one can then start as follows:

Let $x \in A$, if one proves that $r \in A$, then this means that $A$ contains all the limit points and so $A$ is closed.

Example 3.4.1 [1]: The interval $[a, b]$ is closed in $R$

Theorem 3.4.1[1]: A subset $A$ of a metric space $(X, \rho)$ is closed in $X$ if and only if its complement is open in $X$.

Theorem 3.4.2 [1]: The empty set and the whole space $X$ are closed sets.

Proof: Obvious from theorem 3.4.1
Theorem 3.4.3 [5]: let \( \{F_n: n \in \Delta \} \) be a nonempty family of closed sets of a metric space \((X, \rho)\). Then

a. \( \bigcap_{n \in \Delta} F_n \) is closed in (i.e. an arbitrary intersection of closed set is closed)

b. \( \bigcup_{n \in \Delta} F_n \) is in \( X \) (i.e. a finite union of closed set. In \( X \) is closed in \( X \)).

Remark 3.4.1: theorem 3.4.3 (b) does not necessarily hold for infinite union.

3.5 Closure of a set

Let \((X, \rho)\) be a metric space and let \( A \) be a subset of \( X \). The set \( \{x \in X, x \in A \text{ or } x \text{ is a limit point of } A\} \) is called the closure of \( A \) and will be denoted by \((A)\). Hence, \( \overline{A} = A^0 \cup A' \) where we recall \( A \) denotes the set of all limit points of \( A \).

If \( X = \mathbb{R} \) (the real line) with the usual metric and \( A \cup (0,1) \). Then clearly \( \overline{A} = A^0 \cup A' = (0,1) \cup [0,1] = [0,1] \)

Theorem 3.5.1 [1]: A subset \( F \) of a metric space \( X \) is closed if and only if \( F = \overline{F} \)

Theorem 3.5.2 [1]: Every singleton subset of any mother space is closed.

Theorem 3.5.3 [4]: Let \((X, \rho)\) be a metric space. let \( A \) be a subset of \( X \), then a point \( P \) in \( X \) is a limit point of \( A \) if and only if for every real number \( r > 0 \), \( B_r(P) \) contains infinitely many points of \( A \).

3.6 Bounded sets, diameter, boundary

Let \((X, \rho)\) be a metric space and let \( A \) be a subset of \( X \), the diameter of \( A \), denoted by \( \delta(A) \) is defined by

\[
\delta(A) = \sup \{ \rho(x,y): x \in A \}
\]

Where the sup exists. If the sup does not exist, then \( \delta(A): x > \infty \). A subset \( A \) of \( X \) is said to be bounded if \( \delta(A) \) is not infinity. In order to prove that a subset \( A \) of \( X \) is bounded. It is sufficient to prove that in the set \( \{ \rho(x,y): x \in A \text{ and } y \in E \} \) has an upper bound. The existence of a lower bound is obvious. If a subset \( B \) of \( X \) is an unbounded subset of \( X \), then for any positive real number \( \alpha \) however large, we can find point \( a \) and \( b \) in \( B \) such that \( \rho(a, b) > \alpha \).

3.6.1 Boundary

Let \((X, \rho)\) be a metric space and let \( A \) be a subset of \( X \). A point \( \rho \in X \) is [Chidume (1989)] called a boundary point of \( A \) if for every \( \rho > 0 \). The following two conditions are satisfied:

(i) \( B_{\rho}(\rho) \cap A \neq \emptyset \)

(ii) \( B_{\rho} \cap A^c \neq \emptyset \)

The above definition implies that if \( \rho \) is a boundary point of \( A \) then every open ball with center \( \rho \) must contain at least one point of \( A \) and at least one point of \( A^c \). The set of all boundary point of \( A \) is called the boundary of \( A \) and will be denoted by \( \delta A \) or \( Bd(A) \). The boundary of a set is also sometimes called the frontier of the set.

3.7 Subspace of a metric space

Let \((X, \rho)\) be a metric space and \( Y \) be a nonempty subset of \( X \) then [Trench (2010)] the restriction of \( \rho \) to \( Y \times Y \), usually denoted by \( \rho|Y \times Y \) is a metric of \( Y \). This is easy to see because \( \rho \) satisfies all the axioms of a metric for all points of te large set \( X \), so \( \rho \) satisfies these axioms for the subset \( Y \).
Example 3.7.1: Let $X = R$ (the real) with the usual metric $d$. The set $Q$ of rational numbers can be taken as a subspace of $R$.

Remark: Let $Y$ be a subspace of $X$ and let $\rho \in Y$. We now denote the open ball in $X$ with center $\rho$ and radius $x > 0$ by $B_x(\rho, x)$ and the open ball in $Y$ with center $\rho$ and radius $x > 0$ by $B'_x(\rho, X)$. We remark that $B_x(\rho, x)$ and by $(\rho, x)$ are not necessarily the same in fact,

$$B_x(\rho, x) = \{x \in X: \rho(x, y) < x\} \quad \text{where as} \quad B'_x(\rho, \rho) = \{x \in Y: \rho(x, y) < x\}$$

The relation between $B_x(\rho, x)$ and $B'_x(\rho, \rho)$ is given in the following (see Fig. 9).

![Fig. 9. Centre of a metric space](image)

**Theorem 3.7.1 [4]:** let $(X, \rho)$ be a metric and let $(Y, \rho_Y)$ be a subspace of $X$. Let $A$ be a subspace of $Y$ (see Fig. 9).

![Fig. 10. Open subspace of a metric space](image)

$A$ is open in $Y$ and only if, we can find a set $G$ which is of such that $A = Y \cap G$.
Then $A$ is open in $Y$ if and only if there exists a set $G$ which is open in $X$ such that $A = Y \cap G$.

Remark: A set may be open in a subspace $(Y, \rho_Y)$ but not be open in the space $(X, \rho)$, and conversely.

![Diagram: Open subspace of a real metric space IR](image)

**Fig. 11. Open subspace of a real metric space IR**

**Theorem 3.7.2** [4]: Let $(X, \rho_X)$ be a metric space and let and let $(Y, \rho_Y)$ be a subspace of $X$. Let $A$ be a subset of $Y$. Then $A$ is closed in $Y$ if and only if there exists a set $F$ which is closed in $X$ such that $A = Y \cap F, G = X \cap Y$.

**Theorem 3.7.3** [5]: Let $(X, \rho_X)$ be a metric space and $(Y, \rho_Y)$ be a subspace of $X$ then,

1. If every subset $A$ of $Y$ which is open in $Y$ is also open in $X$, then $A$ is open in $X$.
2. If $Y$ is open in $X$ then every subset of $Y$ which is open in $Y$ is open in $X$.

**Theorem 3.7.4** [5]: Let $(X, \rho_X)$ be a metric space and $(Y, \rho_Y)$ be a subspace $X$, then

1. If every subset $A$ of $Y$ which is closed in $Y$ is also closed in $X$, $Y$ is closed in $X$.
2. If $Y$ is closed in $X$, then every subset of $Y$ which is closed in $Y$ is closed in $X$.

**Competing Interests**

Authors have declared that no competing interests exist.

**References**


